

# SIZE AND STANLEY DEPTH OF MONOMIAL IDEALS

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**ABSTRACT.** The Lyubeznik size of a monomial ideal  $I$  of a polynomial ring  $S$  is a lower bound for the Stanley depth of  $I$  decreased by 1. A proof given by Herzog-Popescu-Vladoiu had a gap which is solved here.

*Key words :* Stanley depth, Stanley decompositions, Size, lcm-lattices, Polarization.

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## INTRODUCTION

Let  $S = K[x_1, \dots, x_n]$ ,  $n \in \mathbf{N}$ , be a polynomial ring over a field  $K$  and  $\mathfrak{m} = (x_1, \dots, x_n)$ . Let  $I \supsetneq J$  be two monomial ideals of  $S$  and  $u \in I \setminus J$  a monomial. For  $Z \subset \{x_1, \dots, x_n\}$  with  $(J : u) \cap K[Z] = 0$ , let  $uK[Z]$  be the linear  $K$ -subspace of  $I/J$  generated by the elements  $uf$ ,  $f \in K[Z]$ . A presentation of  $I/J$  as a finite direct sum of such spaces  $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_i K[Z_i]$  is called a *Stanley decomposition* of  $I/J$ . Set  $\text{sdepth}(\mathcal{D}) := \min\{|Z_i| : i = 1, \dots, r\}$  and

$\text{sdepth } I/J := \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I/J\}.$

Let  $h$  be the height of  $a = \sum_{P \in \text{Ass}_S S/I} P$  and  $r$  the minimum  $t$  such that there exist  $\{P_1, \dots, P_t\} \subset \text{Ass}_S S/I$  such that  $\sum_{i=1}^t P_i = a$ . We call the *size* of  $I$  the integer  $\text{size}_S I = n - h + r - 1$ . Lyubeznik [6] showed that  $\text{depth}_S I \geq 1 + \text{size}_S I$ . If Stanley's Conjecture [14] would hold, that is  $\text{sdepth}_S I/J \geq \text{depth}_S I/J$ , then we would get also  $\text{sdepth}_S I \geq 1 + \text{size}_S I$  as it is stated in [4]. Unfortunately, there exists a counterexample in [1] of this conjecture for  $I = S$ ,  $J \neq 0$  and it is possible that there are also counterexamples for  $J = 0$ . However, the counterexample of [1] induces another one for  $J \neq 0$  and  $I \neq S$  generated by 5 monomials, which shows that our result from [9] is tight. This counterexample does not affect Question 1 from [10].

Y.-H. Shen noticed that the second statement of [4, Lemma 3.2] is false when  $I$  is not squarefree and so the proof from [4] of  $\text{sdepth}_S I \geq 1 + \text{size}_S I$  is correct only when  $I$  is squarefree. Since the depth is not a lower bound of  $\text{sdepth}$  due to [1] the lower bound of  $\text{sdepth}$  given by size will have a certain value. The main purpose of this paper is to show the above inequality in general (see Theorem 22).

The important tool in the crucial point of the proof is the application of [5, Theorem 4.5] (a kind of polarization) to the so called the lcm-lattice associated to  $I$  (see [2]). Unfortunately, the polarization does not behaves well with size (see e.g.

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[12, Example 1.2]). Since it behaves somehow better with the so-called bigsize (very different from that introduced in [8], see Definition 3), we have to replace the size with the bigsize. Our bigsize is the right notion for a monomial squarefree ideal  $I \subset S$  (see Theorem 14, an illustration of its proof is given in Examples 15, 17). If  $I$  is not squarefree and  $I^p \subset S^p$  is its polarization then it seems that a better notion will be  $\text{bigsize}_{S^p}(I^p) - \dim S^p + \dim S$ .

The inequality  $\text{sdepth}_S S/I \geq \text{size}_S I$  conjectured in [4] was proved in [15] when  $I$  is squarefree and it is extended in [12]. Our bigsize is useless for this inequality (see Remark 16). A similar inequality is proved by Y.-H. Shen in the frame of the quotients of squarefree monomial ideals [13, Theorem 3.6].

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## 1. SQUAREFREE MONOMIAL IDEALS

The proof of the the following theorem is given in [4] in a more general form, which is correct only for squarefree ideals. For the sake of completeness we recall it here in sketch.

**Theorem 1.** (*Herzog-Popescu-Vladoiu*) *If  $I$  is a squarefree monomial ideal then*

$$\text{sdepth}_S I \geq \text{size}_S(I) + 1.$$

*Proof.* Write  $I = \cap_{i \in [s]} P_i$  as an irredundant intersection of monomial prime ideals of  $S$  and assume that  $P_1 = (x_1, \dots, x_r)$  for some  $r \in [n]$ . Apply induction on  $s$ , the case  $s = 1$  being trivial. Assume that  $s > 1$ . Using [3, Lemma 3.6] we may reduce to the case when  $\sum_{i \in [s]} P_i = \mathfrak{m}$ .

Set  $S' = K[x_1, \dots, x_r]$ ,  $S'' = K[x_{r+1}, \dots, x_n]$ . For every nonempty proper subset  $\tau \subset [s]$  set

$$S_\tau = K[\{x_i : i \in [r], x_i \notin \sum_{j \in \tau} P_j\}],$$

$$J_\tau = (\cap_{i \in [s] \setminus \tau} P_i) \cap S_\tau, \quad L_\tau = (\cap_{i \in \tau} P_i) \cap S''.$$

If  $J_\tau \neq 0$ ,  $L_\tau \neq 0$  define  $A_\tau = \text{sdepth}_{S_\tau} J_\tau + \text{sdepth}_{S''} L_\tau$ . Also define  $A_0 = \text{sdepth}_S I_0$  for  $I_0 = (I \cap S')S$ . By [8, Theorem 1.6] (the ideas come from [7, Proposition 2.3]) we have

$$\text{sdepth}_S I \geq \min\{A_0, \{A_\tau : \emptyset \neq \tau \subset [s], J_\tau \neq 0, L_\tau \neq 0\}\}.$$

Using again [3, Lemma 3.6] we see that if  $I_0 \neq 0$  then  $\text{sdepth}_S I_0 \geq n - r \geq \text{size}_S(I) + 1$ . Fix a nonempty proper subset  $\tau \subset [s]$  such that  $J_\tau \neq 0$ ,  $L_\tau \neq 0$ . It is enough to show that  $A_\tau \geq \text{size}_S(I) + 1$ , that is to verify that  $\text{sdepth}_{S''} L_\tau \geq \text{size}_S(I)$  because  $\text{sdepth}_{S_\tau}(J_\tau) \geq 1$ .

Set  $P_\tau = \sum_{i \in \tau} P_i \cap S''$ , let us say  $P_\tau = (x_{r+1}, \dots, x_e)$  for some  $e \leq n$ . Let  $j_1 < \dots < j_t$  in  $\tau$  with  $t$  minim such that  $\sum_{i=1}^t P_{j_i} \cap S'' = P_\tau$ . Thus  $\text{size}_{S''} L_\tau = t - 1 + n - e$ . Choose  $k_1 < \dots < k_u$  in  $[s] \setminus (\tau \cup \{1\})$  with  $u$  minim such that  $(x_{e+1}, \dots, x_n) \subset \sum_{i=1}^u P_{k_i}$ . We have  $u \leq n - e$ . Then  $P_1 + \sum_{i=1}^t P_{j_i} + \sum_{i=1}^u P_{k_i} = \mathfrak{m}$

and so  $u + t + 1 \geq \text{size}_S(I) + 1$ . By induction hypothesis on  $s$  we have  $\text{sdepth}_{S''} L_\tau \geq \text{size}_{S''} L_\tau + 1 = t + n - e \geq t + u \geq \text{size}_S(I)$ .  $\square$

Now let  $I \subset S$  be a monomial ideal not necessarily squarefree and  $I = \cap_{i \in [s]} Q_i$  an irredundant decomposition of  $I$  as an intersection of irreducible monomial ideals,  $P_i = \sqrt{Q_i}$ . Set  $a = \sum_{i=1}^s P_i$ . Let  $\nu$  be a total order on  $[s]$ . We say that  $\nu$  is *admissible* if given  $i, j, k \in [s]$  with  $j, k > i$  with respect to  $\nu$  and such that from  $\text{height}(\sum_{p \in [i]} P_p + P_k) > \text{height}(\sum_{p \in [i]} P_p + P_j)$  it follows that  $j < k$ . Let  $\mathcal{F} = (Q_{i_k})_{k \in [t]}$  be a family of ideals from  $(Q_j)_{j \in [s]}$ ,  $t \in [s]$ ,  $i_1 < \dots < i_t$  with respect to  $\nu$  such that  $P_{i_k}$  are maximal among  $(P_i)_i$ , and set  $a_{k,\mathcal{F}} = \sum_{j=1}^k P_{i_j} \subset a$ ,  $a_{0,\mathcal{F}} = 0$ ,  $a_{\mathcal{F}} = a_{t,\mathcal{F}}$ ,  $t_{\mathcal{F}} = t$ ,  $h_{\mathcal{F}} = \text{height } a_{\mathcal{F}}$ . Shortly, we speak about a family  $\mathcal{F}$  of  $I$ . If  $I$  is squarefree then each  $P_j$  is maximal among  $(P_i)_i$ .

**Definition 2.** A family  $\mathcal{F}$  of  $I$  with respect to  $\nu$  is *admissible* if  $P_{i_k} \not\subset a_{k-1,\mathcal{F}}$  for all  $k \in [t]$ . The admissible family  $\mathcal{F}$  is *maximal* if  $a_{\mathcal{F}} = a$ , that is, there exist no prime ideal  $P \in \text{Ass}_S S/I$  which is not contained in  $a_{\mathcal{F}}$ .

**Definition 3.** Let  $\mathcal{F}$  be a family of  $I$  with respect to  $\nu$ . If  $t_{\mathcal{F}} = 1$  we set  $\text{bigsize}(\mathcal{F}) = \dim S/P_{i_1}$ . If  $t_{\mathcal{F}} > 1$  then define by recurrence the  $\text{bigsize}(\mathcal{F}) = \min\{\text{bigsize}(\mathcal{F}'), 1 + \text{bigsize}(\mathcal{F}_1)\}$ , where  $\mathcal{F}' = (Q_{i_k})_{1 \leq k < t}$  and  $\mathcal{F}_1$  is the family obtained from the family  $\widetilde{\mathcal{F}}_1 = (Q_{i_t} + Q_{i_k})_{1 \leq k < t}$  removing those ideals  $Q_{i_t} + Q_{i_k}$  which contain another ideal  $Q_{i_t} + Q_{i_{k'}}$  with  $k' \in [t-1] \setminus \{k\}$ . Note that  $\mathcal{F}_1$  is given by  $\text{Ass}_S S/I_1$ , where  $I_1 = \cap_{1 \leq k < t} (Q_{i_t} + Q_{i_k})$ , the decomposition being not necessarily irredundant. Then  $\mathcal{F}_1$  is a family of  $I_1$  with respect to the order induced by  $\nu$  such that roughly speaking  $Q_{i_t} + Q_{i_k}$  is smaller than  $Q_{i_t} + Q_{i_{k'}}$  if  $k < k'$  with respect to  $\nu$ . The integer  $\text{bigsize}(\mathcal{F})$  is called the *bigsize* of  $\mathcal{F}$ . Note that  $\text{bigsize}(\mathcal{F}) \leq t - 1 + \dim S/a_{\mathcal{F}}$ . Set  $\text{bigsize}_{\nu}(I) = \text{bigsize}(\mathcal{F})$  for a maximal admissible family  $\mathcal{F}$  of  $I$  with respect to  $\nu$ . We call the *bigsize* of  $I$  the maximum  $\text{bigsize}_S(I)$  of  $\text{bigsize}_{\nu}(I)$  for all total admissible orders  $\nu$  on  $[s]$ .

**Remark 4.** Note that given a total admissible order  $\nu$  there exists just one maximal admissible family  $\mathcal{F}$  with respect to  $\nu$  so the above definition has sense.

**Example 5.** Let  $n = 6$ ,  $P_1 = (x_1, x_2, x_4)$ ,  $P_2 = (x_1, x_3, x_4, x_6)$ ,  $P_3 = (x_2, x_3, x_4, x_6)$ ,  $P_4 = (x_1, x_4, x_5, x_6)$ ,  $P_5 = (x_1, x_2, x_3, x_5, x_6)$  and set  $I = \cap_{i \in [5]} P_i$ . Then  $\mathcal{F} = \{P_1, P_2, P_5\}$ ,  $\mathcal{G} = \{P_1, P_3, P_4\}$  are maximal admissible families of  $I$  with respect of some total admissible order of  $[5]$ , but  $\text{bigsize}(\mathcal{F}') = \min\{3, 1 + 1\} = 2 = \text{bigsize}(\mathcal{G}')$  and  $\text{bigsize}(\mathcal{F}_1) = 0$ ,  $\text{bigsize}(\mathcal{G}_1) = 1$  which implies  $\text{bigsize}(\mathcal{F}) = 1 < 2 = \text{bigsize}(\mathcal{G})$ . Note that  $a_{k,\mathcal{F}} = a_{k,\mathcal{G}}$  for each  $k \in [3]$ .

**Remark 6.** Assume that  $a_{\mathcal{F}} = (x_1, \dots, x_r)$  for some  $r \in [n]$ . Set  $\tilde{S} = K[x_1, \dots, x_r]$  and let  $\tilde{\mathcal{F}} = (Q_{i_k} \cap \tilde{S})_{k \in [t]}$ . Then  $\text{bigsize}(\mathcal{F}) = n - r + \text{bigsize}(\tilde{\mathcal{F}})$ .

**Remark 7.** Let  $\mathcal{F} = (Q_{i_k})_{k \in [t]}$  be a an admissible family of  $I$  with respect to a total admissible order  $\nu$  and  $r \in [t-1]$ . Then  $\mathcal{G} = (Q_{i_k})_{k \in [r]}$  is an admissible family of  $I$  with respect to  $\nu$  and  $\text{bigsize}(\mathcal{F}) \leq \text{bigsize}(\mathcal{G})$ .

**Remark 8.** Let  $\mathcal{F} = (Q_{i_k})_{k \in [t]}$  be a family of  $I$  with respect to a total admissible order  $\nu$ . Then  $\text{bigsize}(\mathcal{F}) = r - 1 + \dim S / (P_{i_{k_1}} + \dots + P_{i_{k_r}})$  for some  $k_1 < \dots < k_r$  from  $[t]$ .

**Example 9.** Let  $n = 5$ ,  $P_1 = (x_1, x_2)$ ,  $P_2 = (x_2, x_3)$ ,  $P_3 = (x_1, x_4, x_5)$  and  $I = P_1 \cap P_2 \cap P_3$ . Then  $\mathcal{F} = (P_i)_{i \in [3]}$  is a maximal admissible family of  $I$  with respect to the usual order  $\nu$  and  $\text{size}_S I = 1$  because  $P_2 + P_3 = \mathfrak{m}$ . Note that  $\mathcal{F}' = (P_i)_{i=1,2}$  has  $\text{bigsize}(\mathcal{F}') = \min\{3, 1+2\} = 3$  and  $\mathcal{F}_1 = (P_3 + P_i)_{i=1,2}$  has  $\text{bigsize}(\mathcal{F}_1) = 1$ . Thus  $\text{bigsize}(\mathcal{F}) = 2$ .

The order given by  $I = P_2 \cap P_3 \cap P_1$  is not admissible, but the order  $\nu'$  given by  $I = P_2 \cap P_1 \cap P_3$  is admissible. The family  $\mathcal{G} = (P_i)_{i=2,1,3}$  has  $\text{bigsize}(\mathcal{G}') = \min\{3, 1+2\} = 3$  and  $\mathcal{G}_1 = (P_3 + P_i)_{i=2,1}$  has  $\text{bigsize}(\mathcal{G}_1) = 1$ . Thus  $\text{bigsize}(\mathcal{G}) = \min\{3, 1+1\} = 2$ . Similarly, the order  $\nu''$  given by  $\{3, 1, 2\}$  is total admissible, the family  $\mathcal{H} = (P_i)_{i=3,1,2}$  has  $\text{bigsize}(\mathcal{H}') = \min\{2, 1+1\} = 2$  and  $\mathcal{H}_1 = (P_2 + P_i)_{i=3,1}$  has  $\text{bigsize}(\mathcal{H}_1) = 1$ . Thus  $\text{bigsize}(\mathcal{H}) = \min\{2, 1+1\} = 2$  and we have  $\text{bigsize}_{\nu'', S}(I) = 2$ .

**Example 10.** Let  $n = 2$ ,  $Q_1 = (x_1)$ ,  $Q_2 = (x_1^2, x_2)$  and  $I = Q_1 \cap Q_2$ . Then  $P_2$  is the only prime  $P_i$  maximal among  $(P_j)_{j \in [2]}$  and for  $\mathcal{F} = \{P_2\}$  we have  $\text{bigsize}_S(\mathcal{F}) = \text{size}_S(I) = 0$ .

**Example 11.** Let  $n = 4$ ,  $Q_1 = (x_1, x_2^2)$ ,  $Q_2 = (x_2, x_3)$ ,  $Q_3 = (x_3^2, x_4)$  and  $I = Q_1 \cap Q_2 \cap Q_3$ . Then  $\mathcal{F} = (Q_i)_{i \in [3]}$  is a maximal admissible family of  $I$  with respect to the usual order  $\nu$  and  $\text{size}_S I = 1$  because  $P_1 + P_3 = \mathfrak{m}$ . Note that  $\mathcal{F}' = (Q_i)_{i=1,2}$  has  $\text{bigsize}(\mathcal{F}') = \min\{2, 1+1\} = 2$  and  $\mathcal{F}_1 = (Q_3 + Q_i)_{i=1,2}$  has  $\text{bigsize}(\mathcal{F}_1) = 0$ . Thus  $\text{bigsize}(\mathcal{F}) = \min\{2, 1+0\} = 1$ .

The order  $\nu'$  given by  $I = P_2 \cap P_1 \cap P_3$  is admissible. The family  $\mathcal{G} = (Q_i)_{i=2,1,3}$  has  $\text{bigsize}(\mathcal{G}') = 2$  and  $\mathcal{G}_1 = (Q_3 + Q_i)_{i=2,1}$  has  $\text{bigsize}(\mathcal{G}_1) = 0$ . Thus  $\text{bigsize}(\mathcal{G}) = \min\{2, 1+0\} = 1$  and we have  $\text{bigsize}_{\nu', S}(I) = 1$ . Similarly, the order  $\nu''$  given by  $\{2, 3, 1\}$  is total admissible and  $\text{bigsize}_{\nu'', S}(I) = 1$ . Also note that the order  $\bar{\nu}$  given by  $\{3, 2, 1\}$  is total admissible, the family  $\mathcal{H} = (Q_i)_{i=3,2,1}$  has  $\text{bigsize}(\mathcal{H}') = \min\{2, 1+1\} = 2$  and  $\mathcal{H}_1 = (Q_1 + Q_i)_{i=3,2}$  has  $\text{bigsize}(\mathcal{H}_1) = 0$ . Thus  $\text{bigsize}(\mathcal{H}) = \min\{2, 1+0\} = 1$  and we have  $\text{bigsize}_{\bar{\nu}, S}(I) = 1$ .

**Example 12.** Let  $n = 6$ ,  $P_1 = (x_1, x_2)$ ,  $P_2 = (x_1, x_3)$ ,  $P_3 = (x_1, x_6)$ ,  $P_4 = (x_3, x_4)$ ,  $P_5 = (x_3, x_5)$ ,  $P_6 = (x_2, x_4)$ ,  $P_7 = (x_5, x_6)$  and  $I = \cap_{i \in [7]} P_i$ . Let  $\nu$  be the usual order and  $\mathcal{F} = (P_i)_{i \in [5]}$ . Then  $\mathcal{F}$  is maximal admissible and  $\text{bigsize}(\mathcal{F}) = 4 > \text{size}_S I$ . Taking  $\nu'$  given by the order  $\{7, 5, 3, 1, 4\}$  we get a maximal admissible family  $\mathcal{G}$  with  $\text{bigsize}(\mathcal{G}) = 3$ . Thus  $\text{bigsize}_S(I) = 4 > 3 = \text{size}_S I$ .

**Lemma 13.** Let  $\nu$  be a total admissible order on  $[s]$  and  $\mathcal{F} = (Q_{i_k})_{k \in [t]}$  a family of  $I$  with respect to  $\nu$ . Then  $\text{bigsize}(\mathcal{F}) \geq \text{size}_S I$ .

*Proof.* By Remark 8 we have  $\text{bigsize}(\mathcal{F}) = r - 1 + \dim S / (P_{i_{k_1}} + \dots + P_{i_{k_r}})$  for some  $k_1 < \dots < k_r$  from  $[t]$ . We may suppose that  $\sum_{j \in [r]} P_{i_{k_j}} = (x_1, \dots, x_e)$  for some  $e \in [n]$ . Choose for each  $p > e$ ,  $p \leq n$  an  $u_p \in [s]$  such that  $x_p \in P_{u_p}$ . Then  $\sum_{j \in [r]} P_{i_{k_j}} + \sum_{p=e+1}^n P_{u_p} = \mathfrak{m}$  and so  $\text{size } I \leq r - 1 + \dim S / (P_{i_{k_1}} + \dots + P_{i_{k_r}}) = \text{bigsize}(\mathcal{F})$ .  $\square$

Next we present a slightly extension of Theorem 1.

**Theorem 14.** *Let  $I = \cap_{i \in [s]} P_i$  be an irredundant intersection of monomial prime ideals of  $S$ . Then  $\text{sdepth}_S I \geq 1 + \text{bigsize}_S(I)$ .*

*Proof.* Using [3, Lemma 3.6] we may reduce to the case when  $\sum_{i \in [s]} P_i = \mathfrak{m}$ . Apply induction on  $n$ . Assume that  $\text{bigsize}_S(I) = \text{bigsize}(\mathcal{F})$  for a maximal admissible family  $\mathcal{F} = (P_{i_k})_{k \in [t]}$  of  $I$  with respect to a total admissible order  $\nu$ . We may suppose that  $i_t = s$  and  $\sum_{k \in [t-1]} P_{i_k} = (x_{r+1}, \dots, x_n)$ ,  $r \geq 1$ . Set  $S' = K[x_1, \dots, x_r]$ ,  $S'' = K[x_{r+1}, \dots, x_n]$ . We may use [8, Theorem 1.6] even when  $(x_1, \dots, x_r) \notin \text{Ass}_S S/I$  (see [4, Lemma 2.1]). In the notations of Theorem 1 we have

$$\text{sdepth}_S I \geq \min\{A_0, \{A_\tau : \emptyset \neq \tau \subset [s], J_\tau \neq 0, L_\tau \neq 0\}\}.$$

If  $I_0 = (I \cap S')S \neq 0$  then  $A_0 \geq 1 + (n - r) \geq 1 + \dim S/P_{i_t} \geq 2 + \text{bigsize}(\mathcal{F}_1) \geq 1 + \text{bigsize}_S(I)$ .

Now suppose that  $\text{sdepth}_S I \geq A_\tau$  for some  $\tau \subset [s]$  with  $J_\tau \neq 0$ ,  $L_\tau \neq 0$ . Thus  $i_k$  must be in  $\tau$  for any  $k \in [t-1]$  because otherwise  $J_\tau = 0$ . Then  $\mathcal{H} = (P_{i_k} \cap S'')_{k \in [t-1]}$  is a maximal admissible family of  $L_\tau$  with respect to  $\nu$ . Note that  $\text{bigsize}(\mathcal{H}) \geq \text{bigsize}(\mathcal{F}_1)$ . By induction hypothesis on  $n$  we have

$$\text{sdepth}_{S''} L_\tau \geq 1 + \text{bigsize}_{S''}(L_\tau) \geq 1 + \text{bigsize}(\mathcal{H}) \geq 1 + \text{bigsize}(\mathcal{F}_1) \geq \text{bigsize}(\mathcal{F}).$$

Therefore,

$$\text{sdepth}_S I \geq A_\tau \geq 1 + \text{sdepth}_{S''} L_\tau \geq 1 + \text{bigsize}(\mathcal{F}) = 1 + \text{bigsize}_S(I).$$

□

**Example 15.** We illustrate the above proof on the case of  $\mathcal{F}$  given in Example 12. Set  $S' = K[x_5]$ ,  $S'' = K[x_1, x_2, x_3, x_4, x_6]$ . Then  $\tau = \{1, 2, 3, 4, 6\}$  is the only  $\tau \subset [7]$  such that  $J_\tau \neq 0$ . We have  $\text{sdepth}_S I = 5 = 1 + \text{sdepth}_{S''} L_\tau$ . Set  $\tilde{S}' = K[x_4]$ ,  $\tilde{S}'' = K[x_1, x_2, x_3, x_6]$ . Then  $\tilde{\tau} = \{1, 2, 3\}$  is the only  $\tilde{\tau} \subset \tau = [7] \setminus \{5, 7\}$  such that  $J_{\tilde{\tau}} \neq 0$ . We have  $\text{sdepth}_{S''} L_\tau = 4 = 1 + \text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}}$ . Now set  $\hat{S}' = K[x_6]$ ,  $\hat{S}'' = K[x_1, x_2, x_3]$ . Then  $\hat{\tau} = \{1, 2\}$  is the only  $\hat{\tau} \subset \tilde{\tau} = [7] \setminus \{4, 5, 6, 7\}$  such that  $J_{\hat{\tau}} \neq 0$ . We have  $\text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}} = 3 = 1 + \text{sdepth}_{\hat{S}''} L_{\hat{\tau}}$ .

On the other hand,  $\mathcal{H} = \{P_1 \cap S'', P_2 \cap S'', P_3 \cap S'', P_4 \cap S''\}$  is a maximal admissible family of  $L_\tau$  and we have  $\text{bigsize}(\mathcal{H}) = 3 = \text{bigsize}(\mathcal{F}_1)$ . Also note that  $\mathcal{P} = \{P_1 \cap \tilde{S}'', P_2 \cap \tilde{S}'', P_3 \cap \tilde{S}''\}$  is a maximal admissible family of  $L_{\tilde{\tau}}$  and  $\text{bigsize}(\mathcal{P}) = 2 = \text{bigsize}(\mathcal{H}_1)$ . Finally,  $\mathcal{E} = \{P_1 \cap \hat{S}'', P_2 \cap \hat{S}''\}$  is a maximal admissible family of  $L_{\hat{\tau}}$  and  $\text{bigsize}(\mathcal{E}) = 1 = \text{bigsize}(\mathcal{P}_1)$ . Therefore, we have  $\text{sdepth}_S I = 1 + \text{bigsize}(\mathcal{F})$ ,  $\text{sdepth}_{S''} L_\tau = 1 + \text{bigsize}(\mathcal{H})$ ,  $\text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}} = 1 + \text{bigsize}(\mathcal{P})$  and  $\text{sdepth}_{\hat{S}''} L_{\hat{\tau}} = 1 + \text{bigsize}(\mathcal{E})$ .

**Remark 16.** Note that in Example 15 we have  $\text{sdepth}_S S/I = 3 = \text{bigsize}(\mathcal{G}) < 4 = \text{bigsize}(\mathcal{F}) = \text{bigsize}_S(I)$  which shows that the corresponding inequality for  $S/I$  fails using this  $\text{bigsize}$ . As  $\text{sdepth}_{S''} S''/L_\tau = 3$  too, we see that the proof of Theorem 14 fails in the case of the module  $S/I$ . Thus the so-called the splitting of variables for arbitrary  $r$  from [4, Proposition 2.1] does not hold for  $S/I$  (this holds for the case when  $r$  is given by a so-called main prime as it is used in [15]).

**Example 17.** We consider now the case of  $\mathcal{G}$  given in Example 12. Set  $S' = K[x_4]$ ,  $S'' = K[x_1, x_2, x_3, x_5, x_6]$ . Then  $\tau = \{1, 2, 3, 5, 7\}$  is the only  $\tau \subset [7]$  such that  $J_\tau \neq 0$ . We have  $\text{sdepth}_S I = 5 = 1 + \text{sdepth}_{S''} L_\tau$ . Set  $\tilde{S}' = K[x_2]$ ,  $\tilde{S}'' = K[x_1, x_3, x_5, x_6]$ . Then  $\tilde{\tau} = \{3, 5, 7\}$  is the only  $\tilde{\tau} \subset \tau = [7] \setminus \{4, 6\}$  such that  $J_{\tilde{\tau}} \neq 0$ . We have  $\text{sdepth}_{S''} L_\tau = 4 = 1 + \text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}}$ . Now set  $\hat{S}' = K[x_1]$ ,  $\hat{S}'' = K[x_3, x_5, x_6]$ . Then  $\hat{\tau} = \{5, 7\}$  is the only  $\hat{\tau} \subset \tilde{\tau} = [7] \setminus \{2, 4, 6\}$  such that  $J_{\hat{\tau}} \neq 0$ . We have  $\text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}} = 3 = 1 + \text{sdepth}_{\hat{S}''} L_{\hat{\tau}}$ .

On the other hand,  $\mathcal{H} = \{P_7 \cap S'', P_5 \cap S'', P_3 \cap S'', P_1 \cap S''\}$  is a maximal admissible family of  $L_\tau$  and we have  $\text{bigsize}(\mathcal{H}) = 2 = \text{bigsize}(\mathcal{G}_1)$ . Also note that  $\mathcal{P} = \{P_7 \cap \tilde{S}'', P_5 \cap \tilde{S}'', P_3 \cap \tilde{S}''\}$  is a maximal admissible family of  $L_{\tilde{\tau}}$  and  $\text{bigsize}(\mathcal{P}) = 2 > 1 = \text{bigsize}(\mathcal{H}_1)$ . Finally,  $\mathcal{E} = \{P_7 \cap \hat{S}'', P_5 \cap \hat{S}''\}$  is a maximal admissible family of  $L_{\hat{\tau}}$  and  $\text{bigsize}(\mathcal{E}) = 1 = \text{bigsize}(\mathcal{P}_1)$ . Therefore, we have  $\text{sdepth}_S I > 1 + \text{bigsize}(\mathcal{G})$ ,  $\text{sdepth}_{S''} L_\tau > 1 + \text{bigsize}(\mathcal{H})$ ,  $\text{sdepth}_{\tilde{S}''} L_{\tilde{\tau}} = 1 + \text{bigsize}(\mathcal{P})$  and  $\text{sdepth}_{\hat{S}''} L_{\hat{\tau}} = 1 + \text{bigsize}(\mathcal{E})$ .

## 2. BIGSIZE AND STANLEY DEPTH

Let  $I \subset S$  be a monomial ideal and  $I = \bigcap_{i \in [s]} Q_i$  an irredundant decomposition of  $I$  as an intersection of irreducible monomial ideals,  $P_i = \sqrt{Q_i}$ . Let  $G(I)$  be the minimal set of monomial generators of  $I$ . Assume that  $\sum_{P \in \text{Ass}_S S/I} P = \mathfrak{m}$ . Given  $j \in [n]$  let  $\deg_j I$  be the maximum degree of  $x_j$  in all monomials of  $G(I)$ .

**Lemma 18.** *Suppose that  $c := \deg_n I > 1$ , let us say  $c = \deg_n Q_j$  if and only if  $j \in [e]$  for some  $e \in [s]$ . Assume that  $Q_j = (J_j, x_n^c)$  for some irreducible ideal  $J_j \subset S_n = K[x_1, \dots, x_{n-1}]$ ,  $j \in [e]$ . Let  $Q'_j = (J_j, x_n^{c-1}) \subset S$ ,  $Q''_j = (J_j, x_{n+1}) \subset \tilde{S} = S[x_{n+1}]$  and set*

$$\tilde{I} = (\bigcap_{i=e+1}^s Q_i \tilde{S}) \cap (\bigcap_{i \in [e]} Q'_i \tilde{S}) \cap (\bigcap_{i=s+1}^{s+e} Q_i) \subset \tilde{S},$$

where  $Q_i = Q''_{i-s}$  for  $i > s$ , the decomposition of  $\tilde{I}$  being not necessarily irredundant. Then  $\text{sdepth}_{\tilde{S}} \tilde{I} \leq \text{sdepth}_S I + 1$  and  $\text{sdepth}_{\tilde{S}} \tilde{S}/\tilde{I} \leq \text{sdepth}_S S/I + 1$ .

*Proof.* Let  $L_I, L_{\tilde{I}}$  be the LCM-lattices associated to  $I, \tilde{I}$ . The map  $\tilde{S} \rightarrow S$  given by  $x_{n+1} \rightarrow x_n$  induces a surjective join-preserving map  $L_{\tilde{I}} \rightarrow L_I$  and by [5, Theorem 4.5] we get  $\text{sdepth}_{\tilde{S}} \tilde{I} \leq \text{sdepth}_S I + 1$  and  $\text{sdepth}_{\tilde{S}} \tilde{S}/\tilde{I} \leq \text{sdepth}_S S/I + 1$ .  $\square$

With the notations and assumptions of Lemma 18 let

$$\tilde{C} = \{i \in [s] : P_i \tilde{S} \in \text{Ass}_{\tilde{S}} \tilde{S}/\tilde{I}\} \cup ([s+e] \setminus [s]).$$

Choose a total admissible order  $\tilde{\nu}$  on  $\tilde{C}$  and a total admissible order  $\nu$  on  $[s]$  extending the restriction of  $\tilde{\nu}$  to  $[s] \cap \tilde{C}$ . Let  $\tilde{\mathcal{F}} = (\tilde{Q}_{i_k})_{k \in [t]}$  be a family of  $\tilde{I}$  with respect to  $\tilde{\nu}$ . Replace in  $\tilde{\mathcal{F}}$  the ideals  $\tilde{Q}_{i_k}$  by  $Q_{i_k} = \tilde{Q}_{i_k} \cap S$  when  $P_{i_k}$  is maximal in  $\text{Ass}_S S/I$  and  $\tilde{Q}_{i_k}$  is not of the form  $Q'_i \tilde{S}$  or  $Q''_i$  for some  $i \in [e]$ . When  $\tilde{Q}_{i_k}$  is of the form  $Q'_i \tilde{S}$  or  $Q''_i$  for some  $i \in [e]$  then replace in  $\tilde{\mathcal{F}}$  the ideal  $\tilde{Q}_{i_k}$  by  $Q_i$ . If  $P_{i_k}$  is not maximal in  $\text{Ass}_S S/I$  then  $\tilde{Q}_{i_k} \subset Q'_i \tilde{S}$  for some  $i \in [e]$  and we replace in  $\tilde{\mathcal{F}}$  the ideal  $\tilde{Q}_{i_k}$  by  $Q_i$  (this  $i$  is not unique and we have to choose a possible one). Note that  $x_n \in P_{i_k}$  because otherwise  $Q_{i_k} \subset Q_i$  which is impossible.

In this way, we get a family  $\overline{\mathcal{F}}$  of ideals which are maximal in  $\text{Ass}_S S/I$ . Sometimes  $\overline{\mathcal{F}}$  contains the same ideal  $Q_i$ ,  $i \in [e]$  several times. Keeping such  $Q_i$  in  $\overline{\mathcal{F}}$  only the first time when it appears and removing the others we get a family  $\mathcal{F}$  of  $I$  with respect to  $\nu$ .

**Lemma 19.** *There exists a family  $\mathcal{G}$  of  $I$  with respect to  $\nu$  such that  $\text{bigsize}(\tilde{\mathcal{F}}) \geq 1 + \text{bigsize}(\mathcal{G})$ .*

*Proof.* Apply induction on  $t$ . Assume that  $t = 1$ . Then note that  $\text{bigsize}(\tilde{\mathcal{F}}) = \dim \tilde{S}/\tilde{P}_{i_1} = 1 + \dim S/P_{i_1} = 1 + \text{bigsize}(\mathcal{F})$  when  $P_{i_1}$  is maximal in  $\text{Ass}_S S/I$  and  $\tilde{Q}_{i_1}$  is not of the form  $Q'_i \tilde{S}$  or  $Q''_i$  for some  $i \in [e]$ . If  $\tilde{Q}_{i_1} = Q'_i \tilde{S}$  for some  $i \in [e]$  then  $\text{bigsize}(\tilde{\mathcal{F}}) = \dim \tilde{S}/P_i \tilde{S} = 1 + \dim S/P_i = 1 + \text{bigsize}(\mathcal{F})$ . Similarly, it happens when  $\tilde{Q}_{i_1} = Q''_i$  because then  $\dim \tilde{S}/\tilde{P}_{e+i} = \dim S/J_i = 1 + \dim S/P_i$ . If  $\tilde{Q}_{i_1} = Q_l \tilde{S}$  for some  $l \in [s]$  such that  $Q_l \subset Q'_i$  for some  $i \in [e]$  and  $P_l$  is not maximal in  $\text{Ass}_S S/I$  then note that  $\dim \tilde{S}/P_l \tilde{S} = 1 + \dim S/P_l > 1 + \dim S/P_i$ .

Let  $t > 1$ . Assume that  $\text{bigsize}(\tilde{\mathcal{F}}) = t - 1 + \dim \tilde{S}/a_{\tilde{\mathcal{F}}}$ . As above we see that  $\dim \tilde{S}/a_{\tilde{\mathcal{F}}} \geq 1 + \dim S/a_{\mathcal{F}}$ . Let  $\overline{\mathcal{F}} = (Q_{i_k})_{k \in [t]}$ . If  $\overline{\mathcal{F}} = \mathcal{F}$  then we get  $\text{bigsize}(\mathcal{F}) \leq t - 1 + \dim S/a_{\mathcal{F}} \leq \text{bigsize}(\tilde{\mathcal{F}}) - 1$ . Otherwise, assume that  $\mathcal{F} = (Q_{i_k})_{k \in E}$  for some  $E \subsetneq [t]$ . We have  $\text{bigsize}(\mathcal{F}) \leq |E| - 1 + \dim S/a_{\mathcal{F}} < t - 1 + \dim S/a_{\mathcal{F}} \leq \text{bigsize}(\tilde{\mathcal{F}}) - 1$ . Then take  $\mathcal{G} = \mathcal{F}$ .

Now assume that  $\text{bigsize}(\tilde{\mathcal{F}}) = r - 1 + \dim \tilde{S}/\sum_{j \in [r]} \tilde{P}_{i_{k_j}}$  for some  $r \in [t - 1]$  and  $k_1 < \dots < k_r$  from  $[t]$  (see Remark 8). Set  $\tilde{\mathcal{G}} = (\tilde{Q}_{i_{k_j}})_{j \in [r]}$ . We have  $\text{bigsize}(\tilde{\mathcal{G}}) \leq r - 1 + \dim \tilde{S}/a_{\tilde{\mathcal{G}}} = \text{bigsize}(\tilde{\mathcal{F}})$ . Consider the families  $\overline{\mathcal{G}}$ ,  $\mathcal{G}$  corresponding to  $\tilde{\mathcal{G}}$  similarly to  $\overline{\mathcal{F}}$ ,  $\mathcal{F}$  corresponding to  $\tilde{\mathcal{F}}$ . By induction hypothesis ( $r < t$ ) we have  $\text{bigsize}(\tilde{\mathcal{G}}) \geq 1 + \text{bigsize}(\mathcal{G})$ . Then

$$\text{bigsize}(\tilde{\mathcal{F}}) \geq \text{bigsize}(\tilde{\mathcal{G}}) \geq 1 + \text{bigsize}(\mathcal{G}).$$

□

**Example 20.** Let  $n = 4$ ,  $Q_1 = (x_1, x_2)$ ,  $Q_2 = (x_1, x_3)$ ,  $Q_3 = (x_1^2, x_2, x_3)$ ,  $Q_4 = (x_1^2, x_3, x_4)$  and  $I = \cap_{i \in [4]} Q_i$ . Let  $\mathcal{F} = \{Q_3, Q_4\}$ . Then  $\text{size}_S(I) = 1$  because  $P_3 + P_4 = \mathfrak{m}$ . Also note that  $\text{bigsize}(\mathcal{F}') = \min\{1, 1 + 0\} = 1$ ,  $\text{bigsize}(\mathcal{F}_1) = 0$  and so  $\text{bigsize}(\mathcal{F}) = \min\{1, 1 + 0\} = 1$ .

Clearly,  $\tilde{I} = Q_1 \tilde{S} \cap Q_2 \tilde{S} \cap Q_3'' \cap Q_4''$ . Now  $P_1 \tilde{S}, P_2 \tilde{S}$  are maximal in  $\text{Ass}_{\tilde{S}} \tilde{S}/\tilde{I}$ . For  $\mathcal{G} = \{Q_1 \tilde{S}, Q_2 \tilde{S}, Q_3'', Q_4''\}$  we get  $\text{bigsize}(\mathcal{G}') = \min\{\min\{3, 1 + 2\}, 1 + 1\} = 2$ ,  $\text{bigsize}(\mathcal{G}_1) = \min\{1, 1 + 0\} = 1$  and so  $\text{bigsize}(\mathcal{G}) = \min\{2, 1 + 1\} = 2$ . If we take  $\mathcal{H} = \{Q_3'', Q_4'', Q_1\}$  then  $\text{bigsize}(\mathcal{H}') = \min\{2, 1 + 1\} = 2$ ,  $\text{bigsize}(\mathcal{H}_1) = 1$  and so  $\text{bigsize}(\mathcal{H}) = 2$ . Thus  $\text{bigsize}(\mathcal{G}) = \text{bigsize}(\mathcal{H}) = 1 + \text{bigsize}(\mathcal{F})$ .

**Example 21.** Let  $n = 4$ ,  $Q_1 = (x_1, x_2)$ ,  $Q_2 = (x_1^2, x_3)$ ,  $Q_3 = (x_1^2, x_4)$  and  $I = \cap_{i \in [3]} Q_i$ . Let  $\mathcal{F} = \{Q_1, Q_2, Q_3\}$ . Then we see that  $\text{bigsize}(\mathcal{F}) = 2 = \text{size } I$ . Clearly,  $\tilde{I} = Q_1 \tilde{S} \cap Q'_2 \tilde{S} \cap Q'_3 \tilde{S} \cap Q''_2 \cap Q''_3$ , where  $Q'_2 = (x_1, x_3)$ ,  $Q'_3 = (x_1, x_4)$ ,  $Q''_2 = (x_3, x_5)$ ,  $Q''_3 = (x_4, x_5)$ . Then  $\{Q_1 \tilde{S}, Q'_2 \tilde{S}, Q''_3\}$ ,  $\{Q'_2 \tilde{S}, Q_1 \tilde{S}, Q''_3\}$ ,  $\{Q'_3 \tilde{S}, Q_1 \tilde{S}, Q''_2\}$ ,  $\{Q''_2, Q_1 \tilde{S}, Q''_3\}$ ,  $\{Q''_3, Q_1 \tilde{S}, Q'_2 \tilde{S}\}$  are maximal admissible families of  $\tilde{I}$  but with respect to some total orders which are not admissible. However,  $\mathcal{G} = \{Q''_2, Q'_2 \tilde{S}, Q_1 \tilde{S},$

$Q'_3\tilde{S}$  is a maximal admissible family of  $\tilde{I}$  with respect to an admissible order. Note that  $\text{bigsize}(\mathcal{G}') = \min\{3, 1 + 2\} = 3$  and  $\mathcal{G}_1 = \{(x_1, x_3, x_4), (x_1, x_2, x_4)\}$  has  $\text{bigsize}$  2. Thus  $\text{bigsize}(\mathcal{G}) = \min\{3, 1 + 2\} = 3 = 1 + \text{bigsize}(\mathcal{F})$ . We see that  $\text{size}_{\tilde{S}} \tilde{I} = 2$  because  $Q_1 + Q'_2 + Q'_3 = \tilde{\mathbf{m}}$ .

**Theorem 22.** *Let  $I$  be a monomial ideal of  $S$  and  $I = \cap_{i \in [s]} Q_i$  an irredundant decomposition of  $I$  as an intersection of irreducible monomial ideals,  $P_i = \sqrt{Q_i}$ . Then  $\text{sdepth}_S I \geq \text{size}_S I + 1$ .*

*Proof.* Using [3, Lemma 3.6] we may reduce to the case when  $\sum_{P \in \text{Ass}_S S/I} P = \mathbf{m}$ . If  $I$  is squarefree then apply Theorem 1, or Theorem 14 with Lemma 13. Otherwise, assume that  $c = \deg_n I > 1$ . By Lemma 18 there exist  $e$  and a monomial ideal  $\tilde{I}$  such that  $\text{sdepth}_{\tilde{S}} \tilde{I} \leq \text{sdepth}_S I + 1$ . Set  $I^{(1)} = \tilde{I}$  and  $S^{(1)} = \tilde{S}$ . If  $I^{(1)}$  is not squarefree then apply again Lemma 18 for some  $x_i$  with  $\deg_i I^{(1)} > 1$ . We get  $I^{(2)} = (I^{(1)})^{(1)}$ ,  $S^{(2)} = (S^{(1)})^{(1)}$  such that  $S^{(2)} = S[x_{n+1}, x_{n+2}]$ ,  $\text{sdepth}_{S^{(2)}} I^{(2)} \leq \text{sdepth}_S I + 2$ . Applying Lemma 18 by recurrence we get some monomial ideals  $I^{(j)} \subset S^{(j)}$ ,  $j \in [r]$  for some  $r$  such that  $S^{(j)} = S[x_{n+1}, \dots, x_{n+j}]$ ,  $\text{sdepth}_{S^{(j)}} I^{(j)} \leq \text{sdepth}_S I + j$  and  $I^{(r)}$  is a squarefree monomial ideal (thus  $I^{(r)}$  is the polarization of  $I$ ).

Now, let  $\mathcal{F}^{(r)}$  be a maximal admissible family of  $I^{(r)}$  with respect to some total admissible order  $\nu_r$  such that  $\text{bigsize}_{S^{(r)}}(I^{(r)}) = \text{bigsize}_{\nu_r}(I^{(r)}) = \text{bigsize}(\mathcal{F}^{(r)})$ . By Theorem 14 we have  $\text{sdepth}_{S^{(r)}} I^{(r)} \geq 1 + \text{bigsize}(\mathcal{F}^{(r)})$ .

Using Lemma 19 there exists a family  $\mathcal{F}^{(r-1)}$  of  $I^{(r-1)}$  such that  $1 + \text{bigsize}(\mathcal{F}^{(r-1)}) \leq \text{bigsize}(\mathcal{F}^{(r)})$ . Applying again Lemma 19 by recurrence we find a family  $\mathcal{F}$  of  $I$  such that  $r + \text{bigsize}(\mathcal{F}) \leq \text{bigsize}(\mathcal{F}^{(r)})$ . Thus

$$\begin{aligned} \text{sdepth}_S I &\geq \text{sdepth}_{S^{(r)}} I^{(r)} - r \geq \\ \text{bigsize}(\mathcal{F}^{(r)}) - r + 1 &\geq 1 + \text{bigsize}(\mathcal{F}). \end{aligned}$$

Applying Lemma 13 we are done.  $\square$

**Remark 23.** Let  $n = 4$ ,  $P_1 = (x_1, x_2)$ ,  $P_2 = (x_1^2, x_3^2)$ ,  $P_3 = (x_2, x_4)$ ,  $P_4 = (x_3, x_4)$ , and  $J = \cap_{i \in [4]} P_i$ . Note that the polarization of  $J$  is the ideal  $I$  from Examples 12, 15, 17.

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